Testing a novel large-N reduction for $\mathcal{N} = 4$ super Yang-Mills theory on $R \times S^3$

Goro Ishiki,[a,b] **Sang-Woo Kim,**[c] **Jun Nishimura**[b,d] **and Asato Tsuchiya**[e]

[a] *Department of Physics, Osaka University,*
Toyonaka, Osaka 560-0043, Japan

[b] *KEK Theory Center, High Energy Accelerator Research Organization,*
Tsukuba, Ibaraki 305-0801, Japan

[c] *Center for Quantum Spacetime (CQUeST), Sogang University,*
Seoul 121-742, Korea

[d] *Department of Particle and Nuclear Physics,*
Graduate University for Advanced Studies (SOKENDAI),
Tsukuba, Ibaraki 305-0801, Japan

[e] *Department of Physics, Shizuoka University,*
836 Ohya, Suruga-ku, Shizuoka 422-8529, Japan

E-mail: ishiki@post.kek.jp, sangwookim@sogang.ac.kr,
jnishi@post.kek.jp, satsuch@ipc.shizuoka.ac.jp

ABSTRACT: Recently a novel large-N reduction has been proposed as a maximally super-symmetric regularization of $\mathcal{N} = 4$ super Yang-Mills theory on $R \times S^3$ in the planar limit. This proposal, if it works, will enable us to study the theory non-perturbatively on a computer, and hence to test the AdS/CFT correspondence analogously to the recent works on the D0-brane system. We provide a nontrivial check of this proposal by performing explicit calculations in the large-N reduced model, which is nothing but the so-called plane wave matrix model, around a particular stable vacuum corresponding to $R \times S^3$. At finite temperature and at weak coupling, we reproduce precisely the deconfinement phase transition in the $\mathcal{N} = 4$ super Yang-Mills theory on $R \times S^3$. This phase transition is considered to continue to the strongly coupled regime, where it corresponds to the Hawking-Page transition on the AdS side. We also perform calculations around other stable vacua, and reproduce the phase transition in super Yang-Mills theory on the corresponding curved space-times such as $R \times S^3/Z_q$ and $R \times S^2$.

Contents

1 Introduction

The gauge-gravity duality [1] has been one of the most important subjects in string theory over the past decade. The most typical example is the so-called AdS/CFT correspondence between type IIB superstring theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ U(N) super Yang-Mills theory (SYM). Even in this case, however, a complete proof of the duality is still missing[1] partly because it is a strong/weak duality. The region on the string theory side, where (semi-)classical treatments of gravity or string theory is valid, is mapped to the strongly coupled region in the planar large-N limit on the gauge theory side. In order to study $\mathcal{N} = 4$ SYM in the strongly coupled regime from first principles, one needs to have a non-perturbative formulation such as the lattice gauge theory. The problem here is that the supersymmetry algebra includes translational symmetry, which is necessarily broken by the lattice regularization. In order to restore supersymmetry in the continuum limit, one generally has to fine-tune parameters in the lattice action. In fact there are considerable

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[1]See refs. [2, 3] for some attempts to prove the AdS/CFT correspondence by using the worldsheet approach. Also there are remarkable developments based on the integrability [4].

developments in reducing the number of parameters to be fine-tuned,[2] but any lattice formulations of $\mathcal{N} = 4$ SYM proposed so far seem to require fine-tuning at least three parameters [12]. There are also Monte Carlo studies of the $\mathcal{N} = 4$ SYM based on reduction to matrix quantum mechanics of 6 bosonic commuting matrices [13, 14], which confirmed the AdS/CFT correspondence for 1/2 BPS operators.

Here we are aiming at first-principle calculations in $\mathcal{N} = 4$ U(N) SYM respecting supersymmetry maximally. Since we are interested in the planar large-N limit, we may well have a chance to use the idea of the large-N reduction [15]. It asserts that the planar large-N limit of gauge theories can be studied by dimensionally reduced models, which can be obtained by dimensional reduction. The original idea does not work in general because of the spontaneous breaking of the U(1)^D symmetry in the reduced model [16], which led to various proposals [16–22]. Since reduced models can be regularized by making N finite, one may avoid introducing the lattice structure in space-time [18], which causes the breaking of supersymmetry. However, it seemed rather difficult to avoid the problem concerning the instability of the U(1)^D symmetric vacuum without breaking supersymmetry.

Let us recall here that $\mathcal{N} = 4$ U(N) SYM becomes conformally invariant if one sets all the moduli parameters (represented by the expectation values of adjoint scalars) to zero. Hence the theory on R^4 at the conformally invariant point in the moduli space is equivalent to the theory on $R \times S^3$ through conformal mapping. In fact, $R \times S^3$ is obtained as the boundary of AdS_5, when one uses the global coordinate. Thus $\mathcal{N} = 4$ SYM on $R \times S^3$ appears naturally in the context of the AdS/CFT correspondence, for instance, in the so-called pp-wave limit [23–25] and in the bubbling AdS [26–28].

For our purpose it is intriguing to dimensionally reduce the theory by collapsing the S^3 of $R \times S^3$ to a point. The one-dimensional gauge theory obtained in this way is nothing but the plane wave matrix model (PWMM) [23] as pointed out by ref. [29].[3] The PWMM can be regarded as a mass-deformation of the Matrix theory [31] preserving maximal supersymmetry, where the mass parameter corresponds to the curvature of the S^3 before dimensional reduction.[4] The model possesses many classical vacua, all of which preserve maximal supersymmetry.

Recently it has been conjectured [32] that if one picks up a particular classical vacuum of the PWMM, which corresponds to a sequence of fuzzy spheres with different radii, one can actually retrieve the theory before dimensional reduction in the planar limit. (See refs. [33–35] for earlier discussions.) This conjecture may be viewed as a new type of large-N reduction, which extends the original proposal for the flat space-time to a curved one, and at the same time solves the aforementioned problems concerning the vacuum instability. The classical instability is avoided since the PWMM is a massive theory, and the quantum

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[2]See refs. [5–11] for recent works.

[3]In ref. [29] an equivalence between PWMM around the *trivial vacuum* and the *pp-wave limit* of $\mathcal{N} = 4$ SYM on $R \times S^3$ has been shown at one loop in the pure scalar sector. However, discrepancies are found by four-loop calculations [30]. This connection between PWMM and SYM should not be confused with the large-N reduction we are going to discuss.

[4]Originally PWMM appeared as a generalization of the Matrix theory to the pp-wave background [23]. It is also often referred to as the BMN matrix model in the literature. While we obtain formally the same model in the context of large-N reduction, the interpretation of the model is different.

instability is avoided, too, since the vacuum preserves maximal supersymmetry. Since the planar limit is taken in the reduced model, the instanton transition to other vacua is also suppressed and the "fuzziness" of the spheres is removed.

Viewed as a regularization of the $\mathcal{N} = 4$ SYM on $R \times S^3$, the present formulation respects the maximal $SU(2|4)$ supersymmetry (with 16 supercharges) of the PWMM, and in the large-N limit the symmetry is expected to enhance to the full superconformal $SU(2,2|4)$ symmetry, which has 32 supercharges. Considering that the conformal symmetry is broken by any kind of UV regularizations, this regularization is optimal from the viewpoint of preserving supersymmetries.

Let us also emphasize that if one naively regularizes the $\mathcal{N} = 4$ SYM on $R \times S^3$ by introducing an upper bound on the angular momenta on S^3, one necessarily breaks the gauge symmetry as well as supersymmetry. This problem can be dealt with in perturbative calculations by adding appropriate counter-terms [36, 37], but it is not clear what to do with it in nonperturbative calculations that we are aiming at eventually. In the present formulation, the size of the matrices plays the role of the ultraviolet cutoff, which neatly respects both gauge symmetry and supersymmetry.

In this paper we test the novel large-N reduction at weak coupling. In supersymmetric theories, it often occurs that certain properties in the strongly coupled regime remain qualitatively the same in the weakly coupled regime. For instance, the AdS/CFT correspondence at finite temperature [38] suggests that there is a first-order phase transition in the strongly coupled regime of $\mathcal{N} = 4$ SYM on $R \times S^3$ in the planar limit, which corresponds, on the gravity side, to the Hawking-Page transition [39] between the AdS space-time and the AdS black hole. In fact, even in the weak coupling limit of the $\mathcal{N} = 4$ SYM, there exists a first-order deconfinement phase transition [40, 41], which is conjectured to be a continuation of the one in the strongly coupled regime.[5] We confirm the novel large-N reduction at weak coupling by showing that the PWMM indeed reproduces precisely the above phase transition. The main results of this work were reported briefly in our previous publication [44]. As a related work, a test of the large-N reduction has been performed in the high temperature limit up to two-loop [45] (See also ref. [46].). An application of the large-N reduction to $\mathcal{N} = 1$ SYM on $R \times S^3$ was discussed in ref. [47].

This paper is organized as follows. In section 2 we briefly review the deconfinement phase transition in $\mathcal{N} = 4$ SYM on $R \times S^3$ in the weak coupling limit. In section 3 we describe the large-N reduction proposed in ref. [32]. In section 4 we discuss the weak coupling limit of the PWMM around the vacuum corresponding to $R \times S^3$ at finite temperature. In section 5 we show analytically that the critical temperature of the PWMM agrees with that of $\mathcal{N} = 4$ SYM on $R \times S^3$. Derivation of some equations is given in appendix A. In section 6 we show analytically that the free energy of the PWMM agrees with that of $\mathcal{N} = 4$ SYM on $R \times S^3$ under some assumption. In section 7 we perform Monte Carlo simulations to verify this assumption and to demonstrate the agreement of the free energy explicitly. In section 8 we show that $\mathcal{N} = 4$ SYM on a more general space-time $R \times S^3/Z_q$ can be obtained by choosing a different classical vacuum of the same model. In appendix B

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[5]Thermodynamical properties of $\mathcal{N} = 4$ SYM on $R \times S^3$ have been studied also in refs. [42, 43]

we discuss a simpler case of $\mathcal{N} = 8$ SYM on $R \times S^2$, which can be obtained from coinciding fuzzy spheres. Section 9 is devoted to a summary and discussions.

2 Brief review of weakly coupled $\mathcal{N} = 4$ SYM on $R \times S^3$

In this section we briefly review the calculation [40, 41], which showed that weakly coupled $\mathcal{N} = 4$ U(k) SYM on $R \times S^3$ undergoes a deconfinement phase transition in the planar large-k limit at finite temperature. We will see later that equations similar to the ones that appear below are reproduced from the reduced model. In order to make the similarity clearer, we use k instead of N for the gauge group in this section.

Let us introduce a finite temperature T by compactifying the Euclidean time t to a circle with the circumference T^{-1}. Unlike the $T = 0$ case, the holonomy along the t direction becomes nontrivial and it is represented by the holonomy matrix U. One can choose a gauge, in which U takes the diagonal form

$$U = \mathrm{diag}(e^{i\alpha_1}, \ldots, e^{i\alpha_k}), \tag{2.1}$$

where $\alpha_a \in (-\pi, \pi]$ ($a = 1, \ldots, k$) are constant in space-time. The α_a variables are called the gauge field moduli, since they are massless zero modes.

In the weak coupling limit, all the fields except the gauge field moduli can be integrated out at one loop. Since we are going to take the large-k limit, it is convenient to introduce the distribution of the gauge field moduli

$$\rho(\theta) = \frac{1}{k} \sum_{a=1}^{k} \delta(\theta - \alpha_a) . \tag{2.2}$$

The resulting effective theory for $\rho(\theta)$ is given by [40, 41]

$$S = k^2 \int d\theta d\theta' \, \rho(\theta) \, V(\theta - \theta') \, \rho(\theta'), \tag{2.3}$$

$$V(\theta) = \sum_{p=1}^{\infty} \tilde{V}_p \, \cos(p\theta) , \tag{2.4}$$

$$\tilde{V}_p = \frac{1}{p} \Big\{ 1 - 6z_s(x^p) - z_v(x^p) - 4(-1)^{p+1} z_f(x^p) \Big\} , \tag{2.5}$$

where we have introduced dimensionless parameters

$$x = e^{-\beta}, \quad \beta = \frac{1}{R_{S^3} T} \tag{2.6}$$

with R_{S^3} being the radius of S^3. We have also introduced the functions

$$z_s(x) = \frac{x + x^2}{(1 - x)^3}, \quad z_v(x) = \frac{6x^2 - 2x^3}{(1 - x)^3}, \quad z_f(x) = \frac{4x^{\frac{3}{2}}}{(1 - x)^3}, \tag{2.7}$$

which can be interpreted as the single-particle partition functions for the scalars, the vector and the fermions, respectively. Then one can obtain the distribution $\rho(\theta)$ exactly in the large-k limit by solving the saddle-point equation

$$\int_{-\pi}^{\pi} d\theta' \, V'(\theta - \theta') \, \rho(\theta') = 0 \, . \tag{2.8}$$

Obviously the uniform distribution is always a solution to the saddle-point equation. At low temperature, it gives the absolute minimum of the effective action. As a consequence, the center invariance is unbroken and the free energy normalized by $\frac{1}{k^2}$, which we call the *normalized free energy* in what follows, vanishes in the large-k limit. This phase can be interpreted as the confined phase. One can show that there is a first order phase transition at a critical point determined by

$$\tilde{V}_1 = 0 \, , \tag{2.9}$$

which gives $x_c = 7 - 4\sqrt{3}$ [40, 41] in terms of the dimensionless parameter (2.6).

Above the critical temperature, the dominant solution has a compact support $[-\theta_0, \theta_0]$ with $\theta_0 < \pi$, and the equation (2.8) is satisfied only for $\theta \in [-\theta_0, \theta_0]$. The center invariance is broken and the normalized free energy takes a negative value. This phase can be interpreted as the deconfined phase. Near the critical temperature, in particular, the explicit form of $\rho(\theta)$ is given by the Gross-Witten form [40, 41]

$$\rho(\theta) = \begin{cases} \frac{1}{\pi\omega} \left(\cos\frac{\theta}{2}\right) \sqrt{\omega - \sin^2\frac{\theta}{2}} & \text{for } |\theta| \leq \theta_0 \\ 0 & \text{for } |\theta| > \theta_0 \, , \end{cases} \tag{2.10}$$

$$\text{where } \theta_0 = 2\sin^{-1}\sqrt{\omega} \, , \qquad\qquad \omega = 1 - \sqrt{\frac{-\tilde{V}_1}{1 - \tilde{V}_1}} \, . \tag{2.11}$$

The normalized free energy above the critical temperature is obtained for $R_{S^3} = 1$ as [41]

$$\frac{F_{\text{SYM}}}{k^2} = -0.9877(T - T_c) - 4.248(T - T_c)^{\frac{3}{2}} - 11.696(T - T_c)^2 + \mathcal{O}((T - T_c)^{\frac{5}{2}}) \, , \tag{2.12}$$

where $T_c = -1/\ln(7 - 4\sqrt{3}) = 0.37966\cdots$. This phase transition is speculated to be a continuation of the conjectured phase transition at strong coupling, which corresponds to the Hawking-Page transition [39] according to the AdS/CFT correspondence [38].

3   Large-N reduction for $\mathcal{N} = 4$ SYM on $R \times S^3$

In order to regularize $\mathcal{N} = 4$ U(N) SYM on $R \times S^3$ respecting supersymmetry maximally, we use the idea of the large-N reduction. For that we dimensionally reduce the theory by collapsing the S^3 to a point. Thus we obtain a one-dimensional gauge theory

$$S_{\text{PWMM}} = \frac{1}{g^2} \int dt \, \text{tr} \left[\frac{1}{2}(D_t X_M)^2 - \frac{1}{4}[X_M, X_N]^2 + \frac{1}{2}\Psi^\dagger D_t \Psi - \frac{1}{2}\Psi^\dagger \gamma_M [X_M, \Psi] \right.$$
$$\left. + \frac{\mu^2}{2}(X_i)^2 + \frac{\mu^2}{8}(X_a)^2 + i\mu\epsilon_{ijk}X_i X_j X_k + i\frac{3\mu}{8}\Psi^\dagger \gamma_{123}\Psi \right] \, , \tag{3.1}$$

which is nothing but the PWMM [23]. Here the covariant derivative is defined by $D_t = \partial_t - i[A, \cdot]$, where A, as well as X_M and Ψ, is an $N \times N$ matrix depending on t. The range of indices is given by $1 \le M, N \le 9$, $1 \le i, j, k \le 3$ and $4 \le a \le 9$. The model has SU(2|4) supersymmetry, which includes 16 supercharges, and for $\mu = 0$ it reduces to the D0-brane effective theory or the Matrix theory [31].

In fact the model possesses many vacua representing multi fuzzy spheres. Explicitly, they are given by

$$X_i = \mu \bigoplus_{I=1}^{\nu} \Big(L_i^{(n_I)} \otimes \mathbf{1}_{k_I} \Big), \tag{3.2}$$

where $L_i^{(n)}$ are the n-dimensional irreducible representation of the SU(2) generators obeying $[L_i^{(n)}, L_j^{(n)}] = i \epsilon_{ijk} L_k^{(n)}$. The parameters n_I and k_I in (3.2) have to satisfy the relation $\sum_{I=1}^{\nu} n_I k_I = N$. All of these vacua preserve the SU(2|4) supersymmetry, and they are degenerate.

In order to retrieve $\mathcal{N} = 4$ SYM on $R \times S^3$ in the planar limit, one has to pick up a particular background from (3.2), and consider the theory (3.1) around it. Let us consider the case

$$k_I = k, \quad n_I = n + I - \frac{\nu + 1}{2} \qquad \text{for } I = 1, \ldots, \nu \tag{3.3}$$

with odd ν, and take the large-N limit in such a way that

$$n \to \infty, \quad \nu \to \infty, \quad k \to \infty, \quad n - \frac{\nu}{2} \to \infty$$

$$\text{with } \lambda \equiv \frac{g^2 k}{n} \text{ fixed .} \tag{3.4}$$

Then the resulting theory is claimed [32] to be equivalent to the planar limit of $\mathcal{N} = 4$ SYM on $R \times S^3$ with the radius of S^3 and the 't Hooft coupling constant given, respectively, by[6]

$$R_{S^3} = \frac{2}{\mu}, \qquad \lambda_{R \times S^3} = \lambda V_{S^3}, \tag{3.5}$$

where $V_{S^3} = 2\pi^2 (R_{S^3})^3$ is the volume of S^3.

This equivalence may be viewed as an extension of the Eguchi-Kawai equivalence [15] to a curved space-time. It is crucial that we do not need to do anything like momentum quenching [16, 18]. As a consequence, the formulation preserves the SU(2|4) supersymmetry and the gauge symmetry. For a brief review of the equivalence, see ref. [44]. In the following sections, we give a nontrivial test of this proposal by studying the model at weak coupling.

4 Effective theory for the gauge field moduli

Let us first consider a perturbative expansion of the PWMM around the most general background (3.2). The calculation is analogous to what is done in $\mathcal{N} = 4$ SYM on $R \times S^3$

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[6]The relationship between the radius of S^3 and the parameter μ of the PWMM agrees with the one obtained in dimensionally reducing $\mathcal{N} = 4$ SYM on $R \times S^3$ to arrive at the PWMM (3.1).

in section 2. In particular, we introduce finite temperature T in (3.1), and choose a gauge in which the holonomy matrix U is diagonal

$$U = \bigoplus_{I=1}^{\nu} \left(\mathbf{1}_{n_I} \otimes U_I \right), \tag{4.1}$$

$$U_I = \text{diag}\left(e^{i\alpha_1^{(I)}}, \ldots, e^{i\alpha_{k_I}^{(I)}} \right) \qquad \text{for } I = 1, \ldots, \nu, \tag{4.2}$$

where $\alpha_a^{(I)} \in (-\pi, \pi]$ ($a = 1, \ldots, k_I$). In what follows we use the dimensionless parameters β and x as defined in eq. (2.6) with $R_{S^3} = 2/\mu$ anticipating the relationship (3.5).

The effective action for the gauge field moduli around the general background (3.2) is obtained in the weak coupling limit [48]. It can be decomposed into

$$S_{\text{eff}} = 6 S_s + S_v + 4 S_f + S_V, \tag{4.3}$$

where S_s, S_v, S_f and S_V represent the contribution of a scalar, a vector, a fermion and the Vandermonde determinant, respectively. Explicitly, they are given as

$$S_s = \sum_{I,J=1}^{\nu} S_s^{(I,J)}, \quad S_v = \sum_{I,J=1}^{\nu} S_v^{(I,J)}, \quad S_f = \sum_{I,J=1}^{\nu} S_f^{(I,J)}, \quad S_V = \sum_{I=1}^{\nu} S_V^{(I)}, \tag{4.4}$$

$$S_s^{(I,J)} = \frac{1}{2} \sum_{a=1}^{k_I} \sum_{b=1}^{k_J} \sum_{l=|n_I-n_J|/2}^{(n_I+n_J)/2-1} (2l+1) \ln\left\{ 1 + e^{-2\beta(2l+1)} - 2 e^{-\beta(2l+1)} \cos(\alpha_a^{(I)} - \alpha_b^{(J)}) \right\}, \tag{4.5}$$

$$S_v^{(I,J)} = \frac{1}{2} \sum_{a=1}^{k_I} \sum_{b=1}^{k_J} \left[\sum_{l=|n_I-n_J|/2-1+\delta_{IJ}}^{(n_I+n_J)/2-2} (2l+1) \ln\left\{ 1 + e^{-2\beta(2l+2)} - 2 e^{-\beta(2l+2)} \cos(\alpha_a^{(I)} - \alpha_b^{(J)}) \right\} \right.$$
$$\left. + \sum_{l=|n_I-n_J|/2+1}^{(n_I+n_J)/2} (2l+1) \ln\left\{ 1 + e^{-2\beta \cdot 2l} - 2 e^{-\beta \cdot 2l} \cos(\alpha_a^{(I)} - \alpha_b^{(J)}) \right\} \right], \tag{4.6}$$

$$S_f^{(I,J)} = -\frac{1}{2} \sum_{a=1}^{k_I} \sum_{b=1}^{k_J} \left[\sum_{l=|n_I-n_J|/2-1/2}^{(n_I+n_J)/2-3/2} (2l+1) \ln\left\{ 1 + e^{-2\beta(2l+\frac{3}{2})} + 2 e^{-\beta(2l+\frac{3}{2})} \cos(\alpha_a^{(I)} - \alpha_b^{(J)}) \right\} \right.$$
$$\left. + \sum_{l=|n_I-n_J|/2+1/2}^{(n_I+n_J)/2-1/2} (2l+1) \ln\left\{ 1 + e^{-2\beta(2l+\frac{1}{2})} + 2 e^{-\beta(2l+\frac{1}{2})} \cos(\alpha_a^{(I)} - \alpha_b^{(J)}) \right\} \right], \tag{4.7}$$

$$S_V^{(I)} = - \sum_{a \neq b}^{k_I} \ln \left| \sin \frac{\alpha_a^{(I)} - \alpha_b^{(I)}}{2} \right| - (k_I^2 - k_I) \ln 2. \tag{4.8}$$

The second term in (4.8) is needed to make the free energy vanish in the low temperature phase. With this constant term, the total effective action (4.3) agrees exactly with the one presented in ref. [48].

Let us restrict ourselves here to the particular case (3.3), and rewrite the effective action (4.3) in a form analogous to (2.3)–(2.5), which is useful for analytical studies. For

instance, $S_s^{(I,J)}$ in (4.5) can be rewritten as

$$S_s^{(I,J)} = \frac{1}{2} \sum_{a,b=1}^{k} \sum_{l=|n_I-n_J|/2}^{(n_I+n_J)/2-1} (2l+1) \left[\ln\left\{ 1 - e^{-\beta(2l+1)+i(\alpha_a^{(I)}-\alpha_b^{(J)})} \right\} + \text{c.c.} \right]$$

$$= -\sum_{p=1}^{\infty} \frac{1}{p} \sum_{a,b=1}^{k} \cos\left\{ p(\alpha_a^{(I)} - \alpha_b^{(J)}) \right\} \sum_{l=|n_I-n_J|/2}^{(n_I+n_J)/2-1} (2l+1)e^{-p\beta(2l+1)}$$

$$= \sum_{p=1}^{\infty} \frac{1}{p^2} \sum_{a,b=1}^{k} \cos\left\{ p(\alpha_a^{(I)} - \alpha_b^{(J)}) \right\} \frac{\partial}{\partial\beta} \left(\frac{e^{-p\beta(|n_I-n_J|+1)}(1 - e^{-p\beta(n_I+n_J-|n_I-n_J|)})}{1 - e^{-2p\beta}} \right) .$$

We also introduce the distribution function for $\alpha_a^{(I)}$ as

$$\rho^{(I)}(\theta) = \frac{1}{k} \sum_{a=1}^{k} \delta(\theta - \alpha_a^{(I)}) \qquad \text{for } I = 1, \ldots, \nu . \tag{4.9}$$

Then we find that the effective action for the gauge field moduli can be written as

$$S_{\text{eff}} = k^2 \sum_{I,J=1}^{\nu} \int d\theta d\theta' \, \rho^{(I)}(\theta) \, V^{(I,J)}(\theta - \theta') \, \rho^{(J)}(\theta') , \tag{4.10}$$

$$V^{(I,J)}(\theta) = \sum_{p=1}^{\infty} \tilde{V}_p^{(I,J)} \cos(p\theta) , \tag{4.11}$$

$$\tilde{V}_p^{(I,J)} = \frac{1}{p} \left\{ \delta_{IJ} - 6z_s^{(I,J)}(x^p) - z_v^{(I,J)}(x^p) - 4(-1)^{p+1} z_f^{(I,J)}(x^p) \right\} . \tag{4.12}$$

We have introduced the functions

$$z_s^{(I,J)}(x) = x \frac{\partial}{\partial x} \left(\frac{x^{|n_I-n_J|+1}(1 - x^{n_{IJ}})}{1 - x^2} \right) , \tag{4.13}$$

$$z_v^{(I,J)}(x) = x^2 \frac{\partial}{\partial x} \left(\frac{x^{|n_I-n_J|-1+2\delta_{IJ}}(1 - x^{n_{IJ}-2\delta_{IJ}})}{1 - x^2} \right) + \frac{\partial}{\partial x} \left(\frac{x^{|n_I-n_J|+3}(1 - x^{n_{IJ}})}{1 - x^2} \right) , \tag{4.14}$$

$$z_f^{(I,J)}(x) = x^{\frac{3}{2}} \frac{\partial}{\partial x} \left(\frac{x^{|n_I-n_J|+2\delta_{IJ}}(1 - x^{n_{IJ}-2\delta_{IJ}})}{1 - x^2} \right) + x^{\frac{1}{2}} \frac{\partial}{\partial x} \left(\frac{x^{|n_I-n_J|+2}(1 - x^{n_{IJ}})}{1 - x^2} \right) \tag{4.15}$$

with $n_{IJ} = n_I + n_J - |n_I - n_J|$, which can be interpreted as the single-particle partition functions for the scalars, the vector and the fermions, respectively, as in the $\mathcal{N} = 4$ SYM. We can analyze the effective theory (4.10) by the saddle-point method in the large-k limit. The free energy is given by

$$F_{\text{PWMM}} = -TS_{\text{eff}} , \tag{4.16}$$

where S_{eff} is evaluated at the dominant saddle point. We will show in section 6 that

$$\frac{1}{k^2\nu} F_{\text{PWMM}} = \frac{1}{k^2} F_{\text{SYM}} \tag{4.17}$$

in the limit (3.4), where the right-hand side is the normalized free energy of the $\mathcal{N} = 4$ U(k) SYM on $R \times S^3$. Therefore, we define the normalized free energy for the PWMM by the left-hand side of eq. (4.17). The appearance of the $\frac{1}{\nu}$ factor in the above relation is consistent with diagrammatic considerations [32].

Since $\rho^{(I)}(\theta) = \rho^{(I)}(-\theta)$ due to symmetry, $\rho^{(I)}(\theta)$ can be expanded as

$$\rho^{(I)}(\theta) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{p=1}^{\infty} \tilde{\rho}_p^{(I)} \cos(p\theta) . \tag{4.18}$$

In terms of $\tilde{\rho}_p^{(I)}$, the effective action (4.10) is expressed as

$$S_{\text{eff}} = k^2 \sum_{I,J=1}^{\nu} \sum_{p=1}^{\infty} \tilde{\rho}_p^{(I)} \tilde{V}_p^{(I,J)} \tilde{\rho}_p^{(J)} . \tag{4.19}$$

In the low temperature regime, $\tilde{V}_p^{(I,J)}$ given by (4.12) are positive definite matrices. The action (4.19) is therefore minimized by a configuration with $\rho_p^{(I)} = 0$ for all $p \geq 1$, which implies that $\alpha_a^{(I)}$ distribute uniformly. The center invariance is unbroken and the normalized free energy vanishes. This phase can be interpreted as the confined phase. Above the temperature determined by

$$\det \tilde{V}_p^{(I,J)} = 0 \quad \text{for certain } p , \tag{4.20}$$

we obtain a configuration with $\rho_p^{(I)} \neq 0$ for some I and $p \geq 1$, which implies that some of the distributions $\rho^{(I)}(\theta)$ become non-uniform. As a consequence, the center invariance is broken, and the normalized free energy becomes negative. The high temperature phase can be interpreted as the deconfined phase. The distributions $\rho^{(I)}(\theta)$ are determined by the saddle-point equation

$$\sum_{J=1}^{\nu} \int d\theta' V^{(I,J)}(\theta - \theta') \rho^{(J)}(\theta') = 0 \quad \text{for } \theta \in [-\theta_0^{(I)}, \theta_0^{(I)}] \tag{4.21}$$

derived from (4.10), where $[-\theta_0^{(I)}, \theta_0^{(I)}]$ $(\theta_0^{(I)} \leq \pi)$ represents the support of $\rho^{(I)}(\theta)$.

5 Agreement of the critical temperature

In this section we show analytically that the critical temperature of the PWMM agrees with that of $\mathcal{N} = 4$ SYM on $R \times S^3$.

Let us first take the $n \to \infty$ limit in (3.4). Then, the single-particle partition functions (4.13), (4.14) and (4.15) reduce to

$$z_s^{(I,J)}(x) = x \frac{\partial}{\partial x} \left(\frac{x^{|I-J|+1}}{1-x^2} \right) , \tag{5.1}$$

$$z_v^{(I,J)}(x) = x^2 \frac{\partial}{\partial x} \left(\frac{x^{|I-J|-1+2\delta_{IJ}}}{1-x^2} \right) + \frac{\partial}{\partial x} \left(\frac{x^{|I-J|+3}}{1-x^2} \right) , \tag{5.2}$$

$$z_f^{(I,J)}(x) = x^{\frac{3}{2}} \frac{\partial}{\partial x}\left(\frac{x^{|I-J|+2\delta_{IJ}}}{1-x^2}\right) + x^{\frac{1}{2}} \frac{\partial}{\partial x}\left(\frac{x^{|I-J|+2}}{1-x^2}\right). \tag{5.3}$$

Note here that $z_i^{(I,J)}(x)$ $(i = s, v, f)$ and hence $\tilde{V}_p^{(I,J)}$ are $\nu \times \nu$ Toeplitz matrices; i.e.,

$$z_i^{(I+1,J+1)}(x) = z_i^{(I,J)}(x) \quad \text{for } i = s, v, f, \tag{5.4}$$

$$\tilde{V}_p^{(I+1,J+1)} = \tilde{V}_p^{(I,J)}. \tag{5.5}$$

Therefore, we can represent them by $z_i^{(I,J)}(x) = z_i^{(I-J)}(x)$, $\tilde{V}_p^{(I,J)} = \tilde{V}_p^{(I-J)}$ and make a Fourier transformation as

$$\hat{z}_i(x,\lambda) = \sum_{K=-\infty}^{\infty} z_i^{(K)}(x) e^{iK\lambda} \quad \text{for } i = s, v, f, \tag{5.6}$$

$$\hat{V}_p(\lambda) = \sum_{K=-\infty}^{\infty} \tilde{V}_p^{(K)} e^{iK\lambda} = \frac{1}{p}\left\{1 - 6\hat{z}_s(x^p,\lambda) - \hat{z}_v(x^p,\lambda) - 4(-1)^{p+1}\hat{z}_f(x^p,\lambda)\right\}. \tag{5.7}$$

Let $\tilde{v}_p^{(I)}$ be the eigenvalues of $\tilde{V}_p^{(I,J)}$. Then, a theorem for a Toeplitz matrix [49] implies

$$\lim_{\nu \to \infty} \min_I \tilde{v}_p^{(I)} = \min_\lambda \hat{V}_p(\lambda) \tag{5.8}$$

for each p. Since $\hat{V}_p(\lambda)$ is positive in the low temperature phase, (5.8) allows us to replace the condition (4.20) by

$$\min_\lambda \hat{V}_p(\lambda) = 0 \quad \text{for certain } p. \tag{5.9}$$

This can be further replaced by

$$\min_{\lambda, \, p}\left\{p\hat{V}_p(\lambda)\right\} = 0. \tag{5.10}$$

In the appendix A we show that the left-hand side of this equation is given by

$$\min_{\lambda, \, p}\left\{p\hat{V}_p(\lambda)\right\} = \hat{V}_1(0) = \tilde{V}_1, \tag{5.11}$$

where $\tilde{V}_1$ is defined in (2.5). Therefore, the condition (4.20) is indeed equivalent to (2.9), which implies that the critical temperature should agree with $\mathcal{N} = 4$ SYM on $R \times S^3$.

Let us comment that eqs. (5.1), (5.2) and (5.3) actually coincide with the single-particle partition functions for $\mathcal{N} = 8$ SYM on $R \times S^2$ around a specific monopole background. What we have seen in the previous paragraph can therefore be understood as the large-N equivalence between $\mathcal{N} = 4$ SYM on $R \times S^3$ and $\mathcal{N} = 8$ SYM on $R \times S^2$ around the corresponding multi-monopole background [32]. The agreement of the critical temperature for these two theories has been observed numerically in an earlier work [50].

6 Agreement of the free energy

In this section we demonstrate the agreement of the free energy (4.17). For that purpose we pay attention to the single partition functions that appear in the kernel (4.11) of the effective action. We find that $z_i^{(I,J)}(x)$ for the PWMM defined in eqs. (4.13)–(4.15) and $z_i(x)$ for the SYM defined in eq. (2.7) are related to each other as

$$\sum_{J=1}^{\nu} z_i^{(I,J)}(x) = z_i(x) + \Delta z_i^{(I)}(x) \qquad \text{for } i = s, v, f \,, \tag{6.1}$$

$$\text{where} \quad \Delta z_s^{(I)}(x) = -x \frac{\partial}{\partial x} \left\{ \frac{x}{1-x^2} \Delta^{(I)}(x) \right\} \,, \tag{6.2}$$

$$\Delta z_v^{(I)}(x) = -x^2 \frac{\partial}{\partial x} \left\{ \frac{x^{-1}}{1-x^2} \Delta^{(I)}(x) \right\} - \frac{\partial}{\partial x} \left\{ \frac{x^3}{1-x^2} \Delta^{(I)}(x) \right\} \,, \tag{6.3}$$

$$\Delta z_f^{(I)}(x) = -x^{\frac{3}{2}} \frac{\partial}{\partial x} \left\{ \frac{1}{1-x^2} \Delta^{(I)}(x) \right\} - x^{-\frac{1}{2}} \frac{\partial}{\partial x} \left\{ \frac{x^2}{1-x^2} \Delta^{(I)}(x) \right\} \,, \tag{6.4}$$

$$\Delta^{(I)}(x) = \frac{1}{1-x} \left\{ x^I + x^{\nu-I+1} + x^{2n-\nu+I}(1-x^\nu) \right\}. \tag{6.5}$$

For instance, (6.1) for the $i = s$ case can be shown as

$$\sum_{J=1}^{\nu} z_s^{(I,J)}(x) = x \frac{\partial}{\partial x} \left\{ \frac{x}{1-x^2} \sum_{J=1}^{\nu} (x^{|I-J|} - x^{2n-\nu-1+I+J}) \right\} = z_s(x) + \Delta z_s^{(I)}(x) \,. \tag{6.6}$$

Using (6.1), we find that

$$\sum_{J=1}^{\nu} V^{(I,J)}(\theta) = V(\theta) + \Delta V^{(I)}(\theta) \,, \tag{6.7}$$

$$\Delta V^{(I)}(\theta) = -\sum_{p=1}^{\infty} \frac{1}{p} \left\{ 6\Delta z_s^{(I)}(x^p) + \Delta z_v^{(I)} + 4(-1)^{p+1} \Delta z_f^{(I)}(x^p) \right\} \cos(p\theta) \,, \tag{6.8}$$

where $V(\theta)$ is the kernel (2.4) for the $\mathcal{N} = 4$ SYM, and the remaining I-dependent part $\Delta V^{(I)}(\theta)$ decreases exponentially as one moves away from the edges $I = 1$ and $I = \nu$.

Thus we may naturally expect the solution to (4.21) to be

$$\rho^{(I)}(\theta) = \hat{\rho}(\theta) + \Delta \rho^{(I)}(\theta) \,, \tag{6.9}$$

where $\hat{\rho}(\theta)$ is the solution for the $\mathcal{N} = 4$ SYM, and $\Delta \rho^{(I)}(\theta)$ decreases exponentially as one moves away from the edges. We will see such a behavior explicitly by Monte Carlo simulation in section 7. Thus we have

$$\frac{1}{\nu} \sum_{I=1}^{\nu} |\Delta \rho^{(I)}(\theta)| = \mathcal{O} \left(\frac{1}{\nu} \right). \tag{6.10}$$

Substituting (6.9) into (4.10) and using (6.10), we obtain

$$\frac{1}{\nu k^2} S_{\text{eff}} = \int d\theta \, d\theta' \, \hat{\rho}(\theta) \, V(\theta - \theta') \, \hat{\rho}(\theta') + \mathcal{O} \left(\frac{1}{\nu} \right), \tag{6.11}$$

and hence the relationship (4.17) in the limit (3.4).[7]

---

[7]In deriving (4.17), we did not use the condition $n - \nu/2 \to \infty$ in (3.4), which amounts to requiring
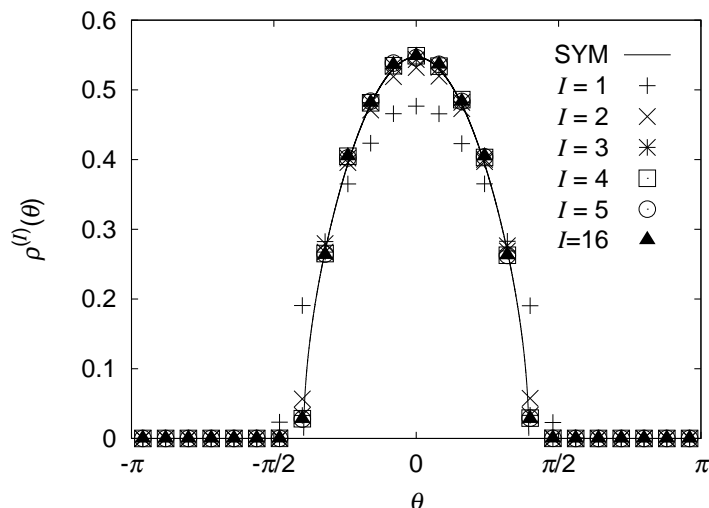
Figure 1. The distribution $\rho^{(I)}(\theta)$ of the gauge field moduli in the PWMM around the background (3.3) is plotted for $I = 1, 2, 3, 4, 5, 16$ with $k = 16$ and $n = \nu = 31$ at a temperature corresponding to $x = 0.104$ near the critical point $x_c \simeq 0.072$. (A similar behavior is obtained for decreasing $I = 31, 30, \ldots, 16$.) The statistical errors are omitted since they are smaller than the symbol size. The solid line represents the result (2.10) for the $\mathcal{N} = 4$ SYM on $R \times S^3$ at the same temperature.

7 Monte Carlo simulation of the effective theory

In this section we confirm the large-N reduction by performing Monte Carlo simulation of the effective theory (4.3) for the gauge field moduli with the particular background (3.3). For the simulation details including the adopted algorithm, we refer the reader to ref. [48], where the effective theory (4.3) for simpler backgrounds was studied. In figure 1 we plot the distribution $\rho^{(I)}(\theta)$ of the gauge field moduli near the critical temperature for various I. As one goes towards the midpoint $I = (\nu + 1)/2$, the distribution converges rapidly to the result (2.10) for the $\mathcal{N} = 4$ SYM on $R \times S^3$ at the same temperature.

Thus we have seen that the distribution of the gauge field moduli has the property described below (6.9). From the argument in the previous section, this implies that the free energy of the PWMM agrees with that of SYM as in eq. (4.17). In order to confirm the agreement more explicitly, we calculate the free energy by using (4.16), where S_{eff} is now replaced by the expectation value of the effective action (4.3) obtained by the Monte Carlo simulation.

We have performed simulations for $k = 40, 100, 200$ and $n = \nu = 15, 23, 31$ at $x = 0.071, 0.074, 0.077$, which are near the critical point $x_c \simeq 0.072$. We first extrapolate our results to $k = \infty$. As is explained in ref. [48], the leading finite k effect for the normalized free energy is given by $\frac{\log k}{k}$, which comes from the first term of (4.8). In figure 2 (left) we plot our Monte Carlo results for the normalized free energy against $\frac{\log k}{k}$ for $n = \nu = 31$. One can see that our data can be nicely fitted by a straight line. This allows us to make

the smallest fuzzy sphere to be regarded as a continuous sphere. It could therefore be that the equivalence holds irrespective of how one takes the large-n and large-ν limits as far as $n - \frac{\nu+1}{2} \geq 0$ is satisfied.
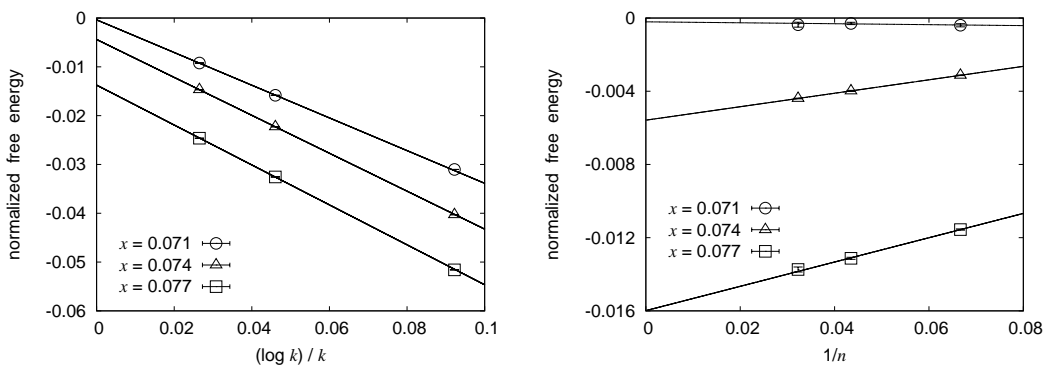
Figure 2. (Left) The normalized free energy is plotted against $\frac{\log k}{k}$ for $n = \nu = 31$. The error bars represent the statistical error. The data can be nicely fitted to a straight line, which enables us to make a large-k extrapolation. (Right) The results after large-k extrapolation are plotted against $1/n$. The error bars represent the fitting error associated with the large-k extrapolation. The data can be nicely fitted to a straight line, which enables us to make a large-n extrapolation.
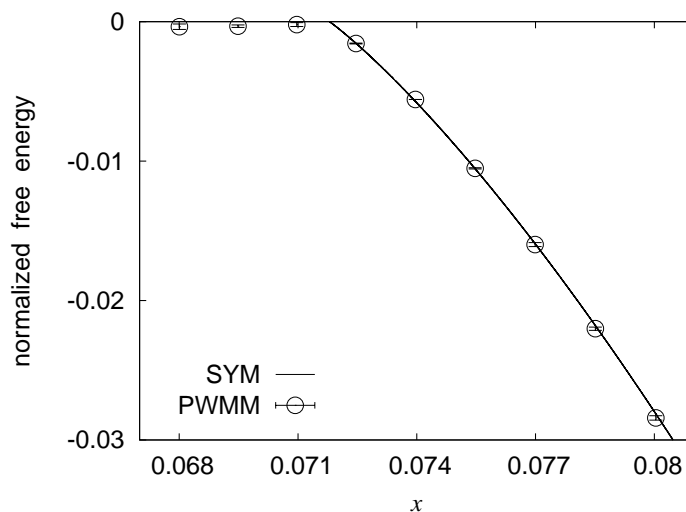

Figure 3. The normalized free energy of the PWMM around the background (3.3) is plotted against the dimensionless parameter x representing temperature near the critical point $x_c = 0.072$. The data points are obtained by extrapolating results for $k = 40, 100, 200$ and $n = \nu = 15, 23, 31$. The error bars represent the fitting error associated with the large-n extrapolation. The solid line represents the result (2.12) for the $\mathcal{N} = 4$ SYM on $R \times S^3$.

an extrapolation to $k = \infty$. In figure 2 (right) we plot the extrapolated values against $1/n$, and find that they lie on a straight line. This allows us to extrapolate our results to $n = \infty$. Thus we obtain the normalized free energy in the limit (3.4), which is plotted in figure 3. Our results obtained from the PWMM agree nicely with the known result (2.12) for the $\mathcal{N} = 4$ SYM on $R \times S^3$.

8   Generalization to $R \times S^3/Z_q$

Our calculation can be generalized to $\mathcal{N} = 4$ SYM on $R \times S^3/Z_q$ ($q \neq 1$) [51] and $\mathcal{N} = 8$ SYM on $R \times S^2$ [52], from which one obtains the PWMM by shrinking S^3/Z_q and S^2 to a point, respectively. Unlike the $\mathcal{N} = 4$ SYM on $R \times S^3$, these SYM theories have only SU(2|4) supersymmetry. They have many classical vacua, all of which preserve this symmetry. The gravity dual corresponding to each vacuum is proposed in ref. [51]. The large-N reduction is expected to work for each vacuum by choosing the corresponding background of the PWMM as one can see from refs. [32–34]. This regularization preserves the full SU(2|4) supersymmetry of the original theory for each vacuum. In what follows we present the calculation for $\mathcal{N} = 4$ SYM on $R \times S^3/Z_q$, which reduces to what we have seen above by setting $q = 1$. A simpler case of $\mathcal{N} = 8$ SYM on $R \times S^2$ is discussed in appendix B. In both cases we restrict ourselves to the trivial vacuum for simplicity.

Let us consider the theory (3.1) around the background (3.2) in the case

$$k_I = k, \ \ n_I = n + q\left(I - \frac{\nu+1}{2}\right) \qquad \text{for } I = 1, \ldots, \nu, \tag{8.1}$$

and take the large-N limit in such a way that[8]

$$n \to \infty, \ \ \nu \to \infty, \ \ k \to \infty, \ \ n - \frac{q\nu}{2} \to \infty$$

$$\text{with } \lambda \equiv \frac{g^2 k}{n} \text{ fixed} . \tag{8.2}$$

The resulting theory is expected to be equivalent to the planar limit of $\mathcal{N} = 4$ SYM on $R \times S^3/Z_q$ [32–34] with the radius of S^3 and the 't Hooft coupling constant given, respectively, by

$$R_{S^3} = \frac{2}{\mu}, \qquad \lambda_{R \times S^3/Z_q} = \lambda \, V_{S^3/Z_q}, \tag{8.3}$$

where $V_{S^3/Z_q} = 2\pi^2 (R_{S^3})^3/q$ is the volume of S^3/Z_q. In what follows we confirm this statement in the weak coupling limit at finite temperature. The effective action is given by the same form as (4.10), where n_I in $V^{(I,J)}$ is given by (8.1).

Let us repeat the calculation in section 5 for general q. We first take the $n \to \infty$ limit in (8.2). Then, (4.13), (4.14) and (4.15) are reduced to

$$z_s^{(I,J)}(x) = x \frac{\partial}{\partial x}\left(\frac{x^{q|I-J|+1}}{1-x^2}\right), \tag{8.4}$$

$$z_v^{(I,J)}(x) = x^2 \frac{\partial}{\partial x}\left(\frac{x^{q|I-J|-1+2\delta_{IJ}}}{1-x^2}\right) + \frac{\partial}{\partial x}\left(\frac{x^{q|I-J|+3}}{1-x^2}\right), \tag{8.5}$$

$$z_f^{(I,J)}(x) = x^{\frac{3}{2}} \frac{\partial}{\partial x}\left(\frac{x^{q|I-J|+2\delta_{IJ}}}{1-x^2}\right) + x^{\frac{1}{2}} \frac{\partial}{\partial x}\left(\frac{x^{q|I-J|+2}}{1-x^2}\right). \tag{8.6}$$

---

[8]Strictly speaking, one has to tune λ as a function of the UV cutoff parameters n and ν according to the coupling constant renormalization in the non-conformal case $q \neq 1$. This issue is irrelevant, though, in the weak coupling limit we are discussing.

Since these are Toeplitz matrices, we can set $z_i^{(I,J)}(x) \equiv z_i^{(I-J)}(x)$ $(i = s, v, f)$. The Fourier transforms (5.6) of these functions are evaluated as

$$\hat{z}_s(x, \lambda) = \frac{x(1 + x^2)(1 - x^{2q})}{(1 - x^2)^2(1 + x^{2q} - 2x^q u)} + \frac{2qx^{q+1}(u - 2x^q + ux^{2q})}{(1 - x^2)(1 + x^{2q} - 2x^q u)^2} \,, \tag{8.7}$$

$$\hat{z}_v(x, \lambda) = \frac{x^2(1 + x^2)}{(1 - x^2)^2} + \frac{2x^q(u - x^q)(q - 1 - (q - 3)x^2) + x^2(3 - x^2)(1 - x^{2q})}{(1 - x^2)^2(1 + x^{2q} - 2x^q u)}$$
$$+ \frac{2qx^{2q}(2u^2 - 1 - 2ux^q + x^{2q}) + 2qx^{q+2}(u - 2x^q + ux^{2q})}{(1 - x^2)(1 + x^{2q} - 2x^q u)^2} \,, \tag{8.8}$$

$$\hat{z}_f(x, \lambda) = \frac{2x^{\frac{5}{2}}}{(1 - x^2)^2} + \frac{2x^{q+\frac{1}{2}}(u - x^q)(2x^2 + q(1 - x^2)) + 2x^{\frac{3}{2}}(1 - x^{2q})}{(1 - x^2)^2(1 + x^{2q} - 2x^q u)}$$
$$+ \frac{2qx^{2q+\frac{1}{2}}(2u^2 - 1 - 2ux^q + x^{2q}) + 2qx^{q+\frac{3}{2}}(u - 2x^q + ux^{2q})}{(1 - x^2)(1 + x^{2q} - 2x^q u)^2} \,, \tag{8.9}$$

where $u = \cos \lambda$. One can easily see that

$$\max_\lambda \hat{z}_s(x, \lambda) = \hat{z}_s(x, 0) = \frac{x(1 + x^2)(1 - x^{2q}) + 2qx^{q+1}(1 - x^2)}{(1 - x^2)^2(1 - x^q)^2} \,,$$

$$\max_\lambda \hat{z}_v(x, \lambda) = \hat{z}_v(x, 0) = \frac{4x^2(1 - x^{2q}) - 2x^q(1 - x^2)^2(1 - x^q) + 2qx^q(1 - x^4)}{(1 - x^2)^2(1 - x^q)^2} \,,$$

$$\max_\lambda \hat{z}_f(x, \lambda) = \hat{z}_f(x, 0) = \frac{2x^{\frac{1}{2}}(1 + x)(x(1 - x^{2q}) + qx^q(1 - x^2))}{(1 - x^2)^2(1 - x^q)^2} \,, \tag{8.10}$$

where the right-hand sides coincide with the single-particle partition functions in SYM on $R \times S^3/Z_q$ obtained at one loop in ref. [53]. By applying the same argument as in appendix A, we find[9]

$$\min_{\lambda, \, p}\left\{p\hat{V}_p(\lambda)\right\} = \hat{V}_1(0) \,. \tag{8.11}$$

Therefore the critical temperature is determined by $\hat{V}_1(0) = 0$, which coincides with the condition for $\mathcal{N} = 4$ SYM on $R \times S^3/Z_q$. Thus we find that the critical temperature in $\mathcal{N} = 4$ SYM on $R \times S^3/Z_q$ is reproduced from PWMM.

We can evaluate the free energy with an assumption made in section 6 and see that it agrees with the continuum theory. We have also performed Monte Carlo simulations of the effective theory for the gauge field moduli as we did for the $q = 1$ case and confirmed the assumption as well as the agreement of the free energy explicitly.

Let us see what happens if we omit fermions and consider a bosonic theory. If we introduce r adjoint scalars instead of 6, the effective action takes the same form as (4.10) except that (4.12) has to be replaced by

$$\tilde{V}_p^{(I,J)} = \frac{1}{p}\left\{\delta_{IJ} - rz_s^{(I,J)}(x^p) - z_v^{(I,J)}(x^p)\right\} \,. \tag{8.12}$$

---

[9]For general q, we had to assume that the critical point lies in the regime $x \ll 1$ to derive (8.11) analytically.

Correspondingly, (5.7) has to be replaced by

$$\hat{V}_p(\lambda) = \sum_{K=-\infty}^{\infty} \tilde{V}_p^{(K)} e^{iK\lambda} = \frac{1}{p}\Big\{ 1 - r\hat{z}_s(x^p, \lambda) - \hat{z}_v(x^p, \lambda) \Big\}, \qquad (8.13)$$

where $\hat{z}_s(x^p, \lambda)$ and $\hat{z}_v(x^p, \lambda)$ are given by (8.7) and (8.8), respectively. From (8.10), we find that (8.11) holds also for (8.13). Therefore, we see that the critical temperature of the bosonic YM theory on $R \times S^3/Z_q$ is reproduced from the corresponding bosonic matrix model. We have also checked the agreement of the free energy.

9 Summary and discussions

In this paper we have provided a nontrivial test of the large-N reduction for $\mathcal{N} = 4$ SYM on $R \times S^3$, which enables us to study the AdS/CFT correspondence from first principles. By expanding the PWMM around a background corresponding to $R \times S^3$, we reproduced the deconfinement phase transition in $\mathcal{N} = 4$ SYM on $R \times S^3$ in the planar limit at weak coupling. The planar $\mathcal{N} = 4$ SYM on $R \times S^3$ can thus be regularized by the PWMM around that background. This regularization preserves the SU(2|4) supersymmetry, which is expected to enhance to the full superconformal SU(2, 2|4) symmetry in the large-N limit. Considering that we are dealing with a finite temperature set-up, which breaks supersymmetry, our results actually suggest that the SO(4) symmetry,[10] which is the bosonic subgroup of the superconformal symmetry, is restored in the large-N limit.

By choosing a different classical vacuum of the same model, we also reproduced the deconfinement transition in SYM on other space-times such as $R \times S^3/Z_q$ and $R \times S^2$. These theories only have SU(2|4) supersymmetry, which is fully preserved by our regularization. The gravity duals are known [51], and if one can calculate various quantities on the gravity side, one can study the gauge-gravity duality from first principles. It is interesting that the PWMM can be used to regularize all these theories in the planar limit by just changing the background configuration.

We were also able to reproduce bosonic theories on curved spaces from the PWMM without fermions around the corresponding backgrounds. However, we consider that supersymmetry is needed to protect the classical backgrounds against radiative corrections at strong coupling. Therefore, we consider that one has to choose sufficiently low temperature or to compactify the Euclidean time direction supersymmetrically in order for the large-N reduction to work at strong coupling.

Monte Carlo simulation of the PWMM can be done in exactly the same way as in the case of D0-brane system, which simply corresponds to the $\mu = 0$ case of the PWMM. In particular, in order to respect supersymmetry maximally, we consider it important not to discretize the Euclidean time direction, but to use finite numbers of Fourier modes after an appropriate gauge fixing [6, 8]. Indeed in the case of D0-brane system, the gauge-gravity duality has been confirmed with high precision including α' corrections [10, 11].

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[10]Restoration of the SO(4) symmetry was also checked by explicit one-loop calculations at zero temperature [32].

The novel large-N reduction discussed in this paper enables us to extend these numerical works to $\mathcal{N} = 4$ SYM on $R \times S^3$ in a straightforward manner. One only has to deform the one-dimensional gauge theory by the mass parameter μ, and prepare an appropriate initial configuration to obtain $R \times S^3$ in the large-N limit. We consider it remarkable that there exists a seemingly feasible way to simulate $\mathcal{N} = 4$ SYM on $R \times S^3$ at strong coupling, and hence to investigate the AdS/CFT correspondence from first principles. Note, in particular, that the existing checks of the AdS/CFT correspondence are restricted to quantities protected by supersymmetry somehow. We hope that our formulation enables us to calculate quantities on the gauge theory side without such restrictions and to compare the results against predictions from the gravity side.

Acknowledgments

A Derivation of eq. (5.11)

In order to derive eq. (5.11) in section 5, let us note first that $\hat{z}_i(x, \lambda)$ in (5.7) can be written explicitly as

$$\hat{z}_s(x, \lambda) = \frac{x - x^3}{(1 + x^2 - 2xu)^2} \,, \tag{A.1}$$

$$\hat{z}_v(x, \lambda) = \frac{2x^2(1 + 2u^2 - 4xu + x^2)}{(1 + x^2 - 2xu)^2} \,, \tag{A.2}$$

$$\hat{z}_f(x, \lambda) = \frac{2x^{\frac{3}{2}}(1 - x)(1 + u)}{(1 + x^2 - 2xu)^2} \,, \tag{A.3}$$

where $u = \cos \lambda$. For instance, (A.1) can be derived as

$$\hat{z}_s(x, \lambda) = x \frac{\partial}{\partial x} \left(\frac{x}{1 - x^2} \left(1 + \sum_{K=1}^{\infty} e^{(\ln x + i\lambda)K} + \sum_{K=1}^{\infty} e^{(\ln x - i\lambda)K} \right) \right) \tag{A.4}$$

$$= x \frac{\partial}{\partial x} \left(\frac{x}{1 + x^2 - 2x \cos \lambda} \right) \,. \tag{A.5}$$

Then one can easily find that

$$\max_{\lambda} \hat{z}_i(x, \lambda) = \hat{z}_i(x, 0) = z_i(x) \quad \text{for } i = s, v, f \,, \tag{A.6}$$

where $z_i(x)$ is defined for the $\mathcal{N} = 4$ SYM in eq. (2.7). For odd p, this implies that

$$\min_{\lambda} \left\{ p\hat{V}_p(\lambda) \right\} = p\hat{V}_p(0) = p\tilde{V}_p \,, \tag{A.7}$$

where $\tilde{V}_p$ is given by eq. (2.5). Taking the derivatives of $z_i(x)$, we find

$$\frac{\partial}{\partial x} z_s(x) = \frac{1 + 4x + x^2}{(1-x)^4} > 0,$$

$$\frac{\partial}{\partial x} z_v(x) = \frac{12x}{(1-x)^4} > 0,$$

$$\frac{\partial}{\partial x} z_f(x) = \frac{6(x^{\frac{1}{2}} + x^{\frac{3}{2}})}{(1-x)^4} > 0,$$ (A.8)

which imply $p\tilde{V}_p > \tilde{V}_1$ for odd $p \neq 1$. Therefore,

$$\min_{\lambda,\, p:\text{odd}} \left\{ p\hat{V}_p(\lambda) \right\} = \hat{V}_1(0) = \tilde{V}_1 .$$ (A.9)

The remaining task is to show that

$$p\hat{V}_p(\lambda) > \tilde{V}_1 \qquad \text{for even } p .$$ (A.10)

Let us note first that

$$\frac{\partial}{\partial u}(p\hat{V}_p) = \frac{-24x^p(x^p - x^{3p}) + 8x^{\frac{3}{2}p}(1 - x^p)(1 + 4x^p + x^{2p}) - 8x^{2p}(1 - x^p)(1 - x^{\frac{1}{2}p})^2 u}{(1 + x^{2p} - 2x^p u)^3} ,$$ (A.11)

from which we find that $p\hat{V}_p(\lambda)$ is minimized either at $u = 1$ or at $u = -1$ for each p. Next, from (A.8) we find that

$$\tilde{V}_1 < 1 - 6z_s(x^p) - z_v(x^p) - 4z_f(x^p) = 1 - \frac{6x^p + 12x^{2p} - 2x^{3p} + 16x^{\frac{3}{2}p}}{(1-x^p)^3} \equiv F_p ,$$ (A.12)

where we have defined a new function F_p of x. Considering that

$$p\hat{V}_p(0) - F_p = 8z_f(x^p) > 0 ,$$

$$p\hat{V}_p(\pi) - F_p = \frac{16x^p(3x^p + 2x^{2p} + 3x^{3p} + x^{\frac{1}{2}p}(1 + x^p)^3)}{(1 - x^{2p})^3} > 0 ,$$ (A.13)

we obtain the inequality (A.10). From (A.9) and (A.10), we obtain (5.11).

B Large-N reduction for $\mathcal{N} = 8$ SYM on $R \times S^2$

In this appendix we consider $\mathcal{N} = 8$ SYM on $R \times S^2$. The "large-N reduction" in this case is nothing but the well-known construction of planar field theories using fuzzy spheres. We discuss it here nevertheless to see how our calculations reduce in this simpler case.

Let us consider the theory (3.1) around the background (3.2) in the case

$$\nu = 1, \ k_1 = k, \ n_1 = n ,$$ (B.1)

and take the large-N limit in such a way that (See footnote 8.)

$$n \to \infty, \ k \to \infty \quad \text{with } \lambda \equiv \frac{g^2 k}{n} \text{ fixed} .$$ (B.2)

The resulting theory is equivalent to the planar limit of $\mathcal{N} = 8$ SYM on $R \times S^2$, with the radius of S^2 and the 't Hooft coupling constant given, respectively, by[11]

$$R_{S^2} = \frac{1}{\mu}\,, \qquad \lambda_{R \times S^2} = \lambda\, V_{S^2}\,, \tag{B.3}$$

where $V_{S^2} = 4\pi (R_{S^2})^2$ is the volume of S^2. After taking the limit (B.2), we find that (4.13), (4.14) and (4.15) are reduced to

$$z_s^{(1,1)}(x) = \frac{x(1 + x^2)}{(1 - x^2)^2}\,, \tag{B.4}$$

$$z_v^{(1,1)}(x) = \frac{4x^2}{(1 - x^2)^2}\,, \tag{B.5}$$

$$z_f^{(1,1)}(x) = \frac{2x^{3/2}(1 + x)}{(1 - x^2)^2}\,, \tag{B.6}$$

where the dimensionless parameter x is defined in eq. (2.6) with $R_{S^3} = 2/\mu$. Rewriting (B.4)–(B.6) in terms of

$$\tilde{x} = \exp\left(-\frac{1}{R_{S^2} T}\right) = \exp\left(-\frac{\mu}{T}\right) = x^2\,, \tag{B.7}$$

they completely agree with the single-particle partition functions in $\mathcal{N} = 8$ SYM on $R \times S^2$ obtained at one loop in ref. [50]. Therefore, the free energy agrees with that of $\mathcal{N} = 8$ SYM on $R \times S^2$.

We can redo the calculation omitting fermions in the PWMM. The free energy of the resulting bosonic matrix model around the background (B.1) agrees in the limit (B.2) with the corresponding bosonic theory on $R \times S^2$.

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[11]The relationship between the radius of S^2 and the parameter μ of the PWMM agrees with the one obtained in dimensionally reducing $\mathcal{N} = 4$ SYM on $R \times S^2$ to arrive at the PWMM (3.1). The fact that R_{S^2} is half the radius of S^3 in (3.5) can be understood by regarding S^3 as an S^1 bundle over S^2 and by dimensionally reducing the S^1 fiber direction to obtain S^2.

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